

PATH-COMPONENT INVARIANTS FOR MODULI SPACES OF POSITIVE SCALAR CURVATURE METRICS

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Abstract: *The Kreck-Stolz s -invariant is a classic path-component invariant for the moduli space of positive scalar curvature metrics. It is an absolute (as opposed to relative) invariant, but this strength comes at the expense of being defined only under restrictive topological conditions. The aim of this paper is to construct an analogous invariant for certain product manifolds on which the s -invariant is not defined.*

§0 Introduction

Given a manifold M which supports a positive scalar curvature metric, an important but difficult question is to determine what can be said about the topology of the space of all positive scalar curvature metrics on M , $\text{Riem}_{scal \geq 0}(M)$. (In this paper we will always assume that spaces of metrics are equipped with the smooth topology.) One can also ask about the topology of the *moduli* space of positive scalar curvature metrics on M . Recall that the diffeomorphism group $\text{Diff}(M)$ acts on the space of all metrics $\text{Riem}(M)$ by pull-back, and this action preserves the property of positive scalar curvature. Thus we can form the moduli space of positive scalar curvature metrics

$$\text{Riem}_{scal \geq 0}(M)/\text{Diff}(M).$$

One can pose analogous questions for other curvature conditions, such as positive or non-negative Ricci curvature, negative sectional curvature etc. There has been much recent activity in this general direction: for example see [BHSW], [BERW], [HSS], [CS], [CM], [Wa1], [Wa2], [Wr1], [Wr2], [BH], [DKT], [FO1-3] and the book [TW].

In this paper we will focus on moduli spaces of positive scalar curvature metrics. However we wish to highlight out at the outset that using the results of [Wr2], all statements involving moduli spaces of positive scalar curvature metrics can be easily modified to yield statements about moduli spaces of non-negative scalar curvature metrics. Although these corresponding results are stronger, we have chosen to focus on positivity in order to simplify the exposition.

A basic tool for studying moduli spaces of positive scalar curvature metrics is the Kreck-Stolz s -invariant (see [KS] or [TW] for details). Under the appropriate topological conditions this allows one to distinguish between different path components of the moduli space of positive scalar curvature metrics. As we will need to refer to these conditions regularly, for convenience we make the following definition:

Definition 0.1. *A closed spin manifold M of dimension $4k - 1$, $k \geq 2$, which admits a positive scalar curvature metric and for which all real Pontrjagin classes vanish will be said to satisfy the Kreck-Stolz conditions.*

It was shown in [KS] that if M satisfies the Kreck-Stolz conditions and (M, g) has positive scalar curvature, then $s(M, g)$ is an invariant of the path-component of positive

scalar curvature metrics containing g . Moreover, if $H^1(M; \mathbb{Z}_2) = 0$ (which ensures that given an orientation for M the spin structure is unique), then $|s|$ can be shown to be an invariant of the path-component of the *moduli space* of positive scalar curvature metrics containing $[g]$. This was used in [KS] to show that the moduli space of positive scalar curvature metrics for any manifold M with $H^1(M; \mathbb{Z}_2) = 0$ satisfying the Kreck-Stolz conditions has infinitely many path-components. (In stark contrast, it was shown in [CM] that the moduli space of positive scalar curvature metrics on closed orientable 3-manifolds is path-connected, provided this space is non-empty.) Using the obvious fact that metrics of positive sectional or positive Ricci curvature also have positive scalar curvature, by examining the underlying space of positive scalar curvature metrics using the s -invariant Kreck and Stolz were able to show that in dimension seven there are manifolds for which the moduli space of positive Ricci curvature metrics has infinitely many path-components, and also examples with positive sectional curvature for which the moduli space of such metrics is not path-connected. The s -invariant has subsequently been used to establish analogous results in other contexts. For example the author showed in [Wr1] that the moduli space of Ricci positive metrics on all homotopy spheres in dimensions $4n - 1 \geq 7$ which bound a parallelisable manifold has infinitely many path-components, showing that this infinite disconnectedness phenomenon occurs through an infinite range of dimensions and providing the first examples away from dimension seven. Very recently, in [DKT] it is shown that in every dimension $4n - 1 \geq 7$, there are infinitely many closed manifolds for which the moduli space of non-negative sectional curvature metrics has infinitely many path-components.

To provide some context, it has long been known (see [LM; IV Theorem 7.7]) that for any closed spin manifold M of dimension $4n - 1 \geq 7$, $\text{Riem}_{\text{scal} \geq 0}(M)$ has infinitely many path-components. It was pointed out in [PS; Remark 2.26] that the same argument used to establish [LM; IV Theorem 7.7] can also be used to show that the corresponding moduli space has infinitely many path-components. In the author's experience this second point is not so widely known. In both cases, the argument can be expressed neatly using Gromov and Lawson's relative index. This is defined for pairs of positive scalar curvature metrics g_0, g_1 on M , and is given by

$$i(g_0, g_1) = \text{ind} D^+(M \times [0, 1], g),$$

where D^+ denotes the Dirac operator and g is any metric on $M \times [0, 1]$ restricting to $dt^2 + g_0$ and $dt^2 + g_1$ in a neighbourhood of the boundary components. It can be shown that this is an invariant of the path-components of positive scalar curvature metrics to which g_0 and g_1 belong, and vanishes if both belong to the same component. The advantage of the Kreck-Stolz s -invariant over this is that it is an *absolute* invariant, i.e. it only depends on a single metric. Indeed $i(g_0, g_1) = s(M, g_0) - s(M, g_1)$ whenever the right-hand side is defined. However Kreck and Stolz show ([KS; 2.16]) that it is not possible to define an absolute invariant of this type without imposing extra topological conditions on M .

The aim of this paper is to demonstrate that it is possible to define an absolute invariant under alternative topological circumstances to those in Definition 0.1. We achieve this by providing an extension of the s -invariant to certain product manifolds. The new setting is as follows: we consider Riemannian product manifolds $(M, g_M) \times (N, g_N)$, where

M satisfies the Kreck-Stolz conditions, g_M has positive scalar curvature, and N is a closed spin manifold of dimension $4l$, $l \geq 1$, with $\hat{A}(N) \neq 0$.

For manifolds in dimensions congruent to 0 modulo 4 the \hat{A} -genus is a topological obstruction to the existence of positive scalar curvature metrics. Nevertheless, any Riemannian product involving a positive scalar curvature metric on one factor can be adjusted by scaling to produce a positive scalar curvature metric. In the above product, there is some very small $c > 0$ such that the metric $c^2 g_M + g_N$ has positive scalar curvature.

We cannot in general use the s -invariant to investigate the moduli space of positive scalar curvature metrics on $M \times N$ however. To see this consider $\hat{A}(N)$. The \hat{A} -genus is a rational linear combination of rational Pontrjagin numbers, and hence if $\hat{A}(N) \neq 0$, this means that some real Pontrjagin class of N is non-zero, and in turn this means that some real Pontrjagin class of $M \times N$ is also non-zero. Thus the Kreck-Stolz conditions are not satisfied by the product $M \times N$ in this case.

Let us summarise our new context in a definition:

Definition 0.2. *A closed Riemannian spin manifold (X, g) with positive scalar curvature will be said to have a Kreck-Stolz product structure if it is isometric to a Riemannian product manifold $(M^{4(k-l)-1}, g_M) \times (N^{4l}, g_N)$, $k - l \geq 2$, $l \geq 1$, where M satisfies the Kreck-Stolz conditions (Definition 0.1), $\hat{A}(N) \neq 0$, and $H^1(X; \mathbb{Z}_2) = 0$.*

Remark: It follows from the Künneth Theorem (see for example [Ha; 3.16]) that the condition $H^1(X; \mathbb{Z}_2) = 0$ forces both $H^1(M; \mathbb{Z}_2) = 0$ and $H^1(N; \mathbb{Z}_2) = 0$. Thus a unique spin structure on X means that the spin structures on M and N are unique. Notice also that the positivity of the scalar curvature of g implies the positivity of the scalar curvature of g_M .

The main result we will establish in this paper as follows:

Theorem 0.3. *Consider a Riemannian manifold (X, g) which has a Kreck-Stolz product structure (Definition 0.2). Then there is a quantity $\tilde{s}(X, g) \in \mathbb{Q}$ such that given any other positive scalar curvature metric g' on X for which (X, g') has a Kreck-Stolz product structure with respect to the same smooth topological product, if g and g' belong to the same path-component of the moduli space of positive scalar curvature metrics on X then $|\tilde{s}(X, g)| = |\tilde{s}(X, g')|$. Moreover, $\text{Riem}_{\text{scal} \geq 0}(X)/\text{Diff}(X)$ has infinitely many path-components of positive scalar curvature metrics distinguished by \tilde{s} .*

Note that if we allowed the degenerate case $l = 0$, i.e. the case where N is a point, \tilde{s} would reduce to s .

To illustrate Theorem 0.3, we will present some explicit examples. Recall that a K3 surface K^4 satisfies $\hat{A}(K^4) = -2$, and so this simply-connected spin manifold cannot support a metric of positive scalar curvature. Similarly there is a simply-connected spin ‘Bott manifold’ B^8 for which $\hat{A}(B^8) = 1$, so this too does not admit a positive scalar curvature metric. (The Bott manifold can be constructed by forming the boundary connected sum of 28 copies of the manifold constructed by plumbing the tangent disk bundle of S^4 to itself according to the E_8 -graph. Thus resulting object has boundary S^7 , and this can then be made into a smooth closed manifold B^8 by gluing in a disc D^8 . Together with $\mathbb{H}P^2$, B^8 generates $\Omega_8^{\text{spin}} \cong \mathbb{Z} \oplus \mathbb{Z}$.) We note that despite the fact that neither K^4 nor B^8 admit positive scalar curvature, both are known to admit Ricci flat metrics. Using Theorem 0.3

together with the definition of \tilde{s} (Definition 2.8) and results from [Wr1; page 2014] we immediately obtain:

Theorem 0.4. *If K^4 denotes the K3 surface, B^8 the Bott manifold, and Σ^{4n-1} is any homotopy n -sphere ($n \geq 2$) which bounds a parallelisable manifold, then there is a sequence g_j of Ricci positive metrics on Σ such that given Ricci flat metrics g_K on K^4 and g_B on B^8 we have*

$$\begin{aligned}\tilde{s}(\Sigma \times K^4, g_j + g_K) &= -\frac{j|bP_{4n}| + q}{2^{2n-3}(2^{2n-1} - 1)}; \\ \tilde{s}(\Sigma \times B^8, g_j + g_B) &= \frac{j|bP_{4n}| + q}{2^{2n-2}(2^{2n-1} - 1)},\end{aligned}$$

where $q = q(\Sigma)$ is an integer depending on Σ , and where bP_{4n} denotes the group of diffeomorphism classes of homotopy spheres bounding a parallelisable manifold of dimension $4n$. In particular, the metrics $g_j + g_K$ respectively $g_j + g_B$ belong to different path-components of the moduli space of positive scalar curvature metrics on $\Sigma \times K^4$ respectively $\Sigma \times B^8$.

Theorem 0.4 should be compared with Theorem 0.7 in [Wr2]. In fact, combining Theorem 0.3 with the results in [Wr2] it is not difficult to show that the condition of positive scalar curvature in Theorem 0.4 can be replaced with non-negative Ricci curvature.

We should also point out that products involving the Bott manifold appear in the stable Gromov-Lawson-Rosenberg conjecture (see for example [S; §1]). Specifically, this claims that the *Rosenberg index* of a connected closed spin manifold M of dimension at least five vanishes if and only if for some $k \geq 1$ the manifold $M \times (B^8)^k$ admits a positive scalar curvature metric. Here $(B^8)^k$ denotes the k -fold product.

The invariant \tilde{s} in Theorem 0.3 is only defined for metrics isometric to a product. Since it takes the same value (up to sign) on any two such metrics in the same path-component of the moduli space of positive scalar curvature metrics, we could regard this as an invariant of the path-component itself, as opposed to merely an invariant of the product metrics within that component. Taking this broader viewpoint raises the following question, which we believe is of independent interest:

Question 0.5. *Consider a product manifold $M \times N$ which admits a positive scalar curvature metric. Can one find conditions on M and N under which every path-component of positive scalar curvature metrics contains a product metric?*

Note that there are situations in which $M \times N$ admits positive scalar curvature metrics but no product metric with positive scalar curvature. For example consider the case where M is a simply-connected spin 4-manifold with $\hat{A}(M) = 0$ which does not admit a positive scalar curvature metric (see [R; Counterexample 1.13]), and where N is a K3 surface. As noted above, a K3 surface is a simply-connected spin manifold with non-zero \hat{A} -genus, and therefore does not support a metric of positive scalar curvature. The product $M \times N$ is then a simply-connected spin 8-manifold with $\hat{A}(M \times N) = \hat{A}(M)\hat{A}(N) = 0$. By Gromov-Lawson [GL], all simply-connected spin 8-manifolds with vanishing \hat{A} -genus admit positive scalar curvature metrics. The author is grateful to Boris Botvinnik for pointing out this example.

This paper is laid out as follows. In §1 we outline the construction of the Kreck-Stolz s -invariant, as this provides the blueprint for establishing Theorem 0.3. The construction of \tilde{s} and the proof of Theorem 0.3 are contained in §2.

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§1 The Kreck-Stolz s -invariant

We begin by recalling the index theorem of Atiyah-Patodi-Singer:

Theorem 1.1. ([APS]) *Let (W, g_W) be a compact even dimensional Riemannian spin manifold with non-empty boundary M , where the metric g_W is a product $dt^2 + g_M$ in a neighbourhood of the boundary. Consider the Atiyah-Singer Dirac operator D^+ on W acting on the subspace of spinor bundle sections for which the restriction to M belongs to the span of the negative eigenspaces of the operator induced on M . Then the index of this (restricted domain) Dirac operator on W is given by*

$$\text{ind} D^+(W, g_W) = \int_W \hat{A}(p_*(W, g_W)) - \frac{h(M, g_M) + \eta(M, g_M)}{2},$$

where \hat{A} denotes the \hat{A} -polynomial in the Pontrjagin forms of the metric, h is the dimension on the space of harmonic spinors on the boundary M , and η is the eta-invariant of the Dirac operator on M .

Later on we will encounter eta-invariants of other operators. We will adopt the convention that if an operator is not specified in the notation it should be assumed to be a Dirac operator.

The integral term appearing in Theorem 1.1 appears to depend both on the topology of W and the metric on W . However it is not difficult to see that only the metric in a neighbourhood of the boundary actually influences the value of the integral. (The argument behind this is detailed in Lemma 2.5.) In the light of this observation it is natural to ask: can we separate out the topological dependence on W from the metric dependence near the boundary? The answer to this is a qualified yes: the integral of any summand in the integrand can be rewritten in this way provided it is decomposable (in the sense that it involves a product of forms), and provided that M has vanishing real Pontrjagin classes.

From now on let us assume that M does indeed have vanishing real Pontrjagin classes. Let α, β denote Pontrjagin forms or products of Pontrjagin forms on W . As a consequence of the above assumption together with the product structure of the metric g_W near the boundary, following the notation in [KS; 2.8] we can define a form $d^{-1}(\alpha \wedge \beta)$ on M by setting $d^{-1}(\alpha \wedge \beta) = \hat{\alpha} \wedge (\beta|_M)$, where $\hat{\alpha}$ satisfies $d\hat{\alpha} = \alpha|_M$. A simple Stokes' Theorem argument then shows that

$$\int_W \alpha \wedge \beta = \int_M d^{-1}(\alpha \wedge \beta) + \langle j^{-1}[\alpha] \cup j^{-1}[\beta], [W, M] \rangle,$$

where $j : H^*(W, M; \mathbb{R}) \rightarrow H^*(W; \mathbb{R})$ is the map induced by inclusion, and the angled brackets denote evaluation on the fundamental homology class. By the cohomology long exact sequence of the pair (W, M) it is easy to see that we need M to have vanishing real Pontrjagin classes in order for the required pre-images under j to be defined.

Suppose now that W has dimension $4k$. The top-dimensional term of the \hat{A} -polynomial has all its summands decomposable except for the term in $p_k(W, g_W)$. In order to deal with this, a linear combination of the \hat{A} and L -polynomials is formed which has zero p_k term. Specifically, Theorem 1.1 is applied to $\hat{A} + a_k L$ where $a_k = 1/(2^{2k+1}(2^{2k-1} - 1))$. The resulting index formula is given by

$$\begin{aligned} \text{ind} D^+(W, g_W) = & \int_M d^{-1}(\hat{A} + a_k L)(p_*(M, g_M)) \\ & - \frac{h(M, g_M) + \eta(M, g_M)}{2} \\ & - a_k \eta(B(M, g_M)) - t(W), \end{aligned}$$

where $\eta(B(M, g_M))$ is the eta-invariant of the signature operator B on M , and $t(W)$ is the topological term

$$t(W) = -\langle (\hat{A} + a_k L)(j^{-1} p_*(W)), [W, M] \rangle + a_k \sigma(W)$$

where the $p_*(W)$ are the Pontrjagin classes of W (as opposed to forms), and $\sigma(W)$ is the signature. If we assume that the scalar curvature of M is positive, this forces $h(M, g_M) = 0$.

The idea behind the s -invariant is to collect together all the terms depending on the boundary (M, g_M) :

Definition 1.3. *Given a closed spin manifold M^{4k-1} with positive scalar curvature and vanishing real Pontrjagin classes, the s -invariant is given by*

$$s(M, g) = -\frac{1}{2} \eta(M, g_M) - a_k \eta(B(M, g_M)) + \int_M d^{-1}(\hat{A} + a_k L)(p_*(M, g_M)).$$

With this definition we see immediately that

$$s(M, g_M) = \text{ind} D^+(W, g_W) + t(W).$$

Moreover if the metric g_W also has positive scalar curvature then the index term above vanishes, leaving $s(M, g_M) = t(W)$. In this situation s is completely determined by the topology of the bounding manifold W .

Note that Definition 1.3 does not require M to be the boundary of a suitable manifold W .

The key properties of the s -invariant can be proved by applying the above analysis to the case $W = M \times I$ for any interval I , after showing that s is additive across disjoint unions and is sign-sensitive to orientation. Specifically it can be shown (as already mentioned in §0) that s is a path-component invariant for the space of positive scalar curvature metrics. Furthermore if $H^1(M; \mathbb{Z}_2) = 0$ then $|s(M, g)| \in \mathbb{Q}$ is an invariant of the path-component

of the moduli space of positive scalar curvature metrics on M containing g . (See [KS; Proposition 2.13].)

§2 An extension of the Kreck-Stolz s -invariant

Let $W^{4(k-l)}$ ($k > l \geq 1$) and N^{4l} be compact spin manifolds. We suppose that W has boundary M (possibly disconnected), and that N is a closed manifold. If π_W (respectively π_N) denote the projections of $W \times N$ onto W (respectively N), then by the Whitney formula the total rational or real Pontrjagin class satisfies $p(W \times N) = p(\pi_W^*(TW))p(\pi_N^*(TN))$ since $T(W \times N) \cong \pi_W^*(TW) \oplus \pi_N^*(TN)$. By the multiplicative property of the \hat{A} -polynomial we obtain the equality of polynomials $\hat{A}(p(W \times N)) = \hat{A}(p(\pi_W^*(TW)))\hat{A}(p(\pi_N^*(TN)))$. By the naturality of the Pontrjagin classes we can then write

$$\hat{A}(p(W \times N)) = \pi_W^*(\hat{A}(p(M)))\pi_N^*(\hat{A}(p(N))).$$

Lemma 2.1. *With W, N as above, suppose we choose a product metric $g_W + g_N$ on $W \times N$. Then for the Pontrjagin forms corresponding to this metric we have*

$$p_i(W \times N; g_W + g_N) = \sum_{j+k=i} \pi_W^* p_j(W; g_W) \wedge \pi_N^* p_k(N; g_N).$$

Proof. The Pontrjagin forms are symmetric polynomials in the curvature form for the given metric. Recall that given a local tangent frame field $\{s_i\}$ for a Riemannian n -manifold, the curvature form Ω is an $(n \times n)$ -matrix of 2-forms (Ω_j^i) with entries defined by

$$R(X, Y)(s_j) = \sum_{i=1}^n \Omega_j^i(X, Y)s_i.$$

For a product metric such as $g_W + g_N$ and frame fields $s_1, \dots, s_{4(k-l)} \in \Gamma(TW \oplus 0) \subset T(W \times N)$ and $s_{4(k-l)+1}, \dots, s_{4k} \in \Gamma(0 \oplus TN)$, the curvature 2-form satisfies

$$\Omega = \begin{pmatrix} \Omega_W & 0 \\ 0 & \Omega_N \end{pmatrix}$$

where Ω_W and Ω_N are the pull-backs of the curvature forms of (W, g_W) respectively (N, g_N) . (See [Mo] page 208.) The total Pontrjagin form is then given by

$$\det\left(\mathbb{I} - \frac{1}{2\pi i}\Omega\right) = \det\left(\mathbb{I} - \frac{1}{2\pi i}\Omega_W\right) \wedge \det\left(\mathbb{I} - \frac{1}{2\pi i}\Omega_N\right),$$

where the determinants on the right-hand side are the pull-backs (to $W \times N$) of the total Pontrjagin forms of (W, g_W) and (N, g_N) . The lemma then follows by expanding these total classes into their individual terms. \square

Corollary 2.2. *With the set-up of Lemma 2.1 we have the following decomposition of \hat{A} -polynomials into Pontrjagin forms:*

$$\hat{A}(p_*(W \times N; g_W + g_N)) = \pi_W^* \hat{A}(p_*(W; g_W)) \wedge \pi_N^* \hat{A}(p_*(N; g_N)).$$

In particular for the top-dimensional forms we have

$$\hat{A}_k(p_*(W \times N; g_W + g_N)) = \pi_W^* \hat{A}_{k-l}(p_*(W; g_W)) \wedge \pi_N^* \hat{A}_l(p_*(N; g_N)).$$

Proof. As discussed above, the equivalent formula to the first statement holds for Pontrjagin classes, and follows from the decomposition of those classes on product manifolds. By Lemma 2.1 Pontrjagin forms for product manifolds equipped with product metrics decompose into terms involving the individual factor manifolds in exactly the same way as Pontrjagin classes. The result follows immediately. For the second statement we simply note that a top dimensional form on $W \times N$ can only be formed from a product of top dimensional forms on the factors, since any higher degree form must be zero. \square

Lemma 2.3. *Given top-dimensional differential forms α, β on oriented manifolds X respectively Y , we have*

$$\int_{X \times Y} \pi_X^* \alpha \wedge \pi_Y^* \beta = \left(\int_X \alpha \right) \left(\int_Y \beta \right).$$

Proof. This equation holds as it holds locally in any coordinate neighbourhood which is a product of coordinate neighbourhoods for X and Y individually. Using such a coordinate system the calculation reduces to showing that for appropriate functions a and b :

$$\begin{aligned} & \int_{U \times V} a(x_1, \dots, x_r) b(y_1, \dots, y_s) dx_1 \dots dx_r dy_1 \dots dy_s \\ &= \int_U a(x_1, \dots, x_r) dx_1 \dots dx_r \int_V b(y_1, \dots, y_s) dy_1 \dots dy_s, \end{aligned}$$

which holds trivially. \square

Corollary 2.4. *With W, N and metrics as above we have*

$$\int_{W \times N} \hat{A}_k(p_*(W \times N; g_W + g_N)) = \hat{A}(N) \int_W \hat{A}_{k-l}(p_*(W; g_W)),$$

where $\hat{A}(N)$ is the \hat{A} -genus of N .

Proof. This follows immediately from Corollary 2.2, Lemma 2.3 and the fact that $\hat{A}(N) = \int_N \hat{A}_l(p_*(N; g_N))$.

From now on we will assume that the product metric $g_W + g_N$ takes the form $dt^2 + g_M + g_N$ near the boundary. Since the Pontrjagin forms are defined using the Levi-Civita connection of the metric, if we replace the $g_W + g_N$ by another metric g which takes the same product form $dt^2 + g_M + g_N$ near the boundary we will change the Pontrjagin forms, however this will not change the value of the integral over $W \times N$:

Lemma 2.5. *Consider an oriented manifold X^{4n} with non-empty connected boundary Y . Let ϕ be a top dimensional Pontrjagin form (respectively a top dimensional wedge product of Pontrjagin forms) on X corresponding to a Riemannian metric g_X , which is a product $dt^2 + g_Y$ near the boundary. If ϕ' is the top dimensional Pontrjagin form (respectively the corresponding wedge product of Pontrjagin forms) arising from a metric g'_X which also takes the form $dt^2 + g_Y$ near the boundary, then*

$$\int_X \phi = \int_X \phi'.$$

Proof. We consider the metric $g_X \cup g'_X$ on the oriented double of X , $X \cup (-X)$. This metric is smooth as the individual metrics agree near the common boundary. Now all oriented double manifolds are oriented boundaries, and hence all Pontrjagin numbers of $X \cup (-X)$ must vanish. Thus if we let ψ be the top dimensional Pontrjagin form (respectively wedge product of Pontrjagin forms) on $X \cup (-X)$ arising from $g_X \cup g'_X$, then

$$\int_{X \cup (-X)} \psi = 0.$$

But

$$\begin{aligned} \int_{X \cup (-X)} \psi &= \int_X \phi + \int_{-X} \phi' \\ &= \int_X \phi - \int_X \phi'. \end{aligned}$$

Thus $\int_X \phi = \int_X \phi'$ as claimed. \square

From Corollary 2.4 and Lemma 2.5 we obtain:

Corollary 2.6. *Let W , N , g_W and g_N be as before, with g_W taking the form $dt^2 + g_M$ near $\partial W = M$, and let g be any metric on $W \times N$ which takes the same product form $dt^2 + g_M + g_N$ as $g_W + g_N$ near the boundary. Then*

$$\int_{W \times N} \hat{A}_k(p_*(W \times N; g)) = \hat{A}(N) \int_W \hat{A}_{k-l}(p_*(W; g_W)),$$

and applying the Atiyah-Patodi-Singer index theorem to $(W \times N; g)$ we obtain

$$\text{ind} D_{(W \times N; g)}^+ = \hat{A}(N) \int_W \hat{A}_{k-l}(p_*(W; g_W)) - \frac{h + \eta}{2}(M \times N; g_M + g_N).$$

Following [KS], from now on we will make the assumption that the real Pontrjagin classes of $M = \partial W$ vanish. This assumption allows us to re-write the above integral. Following the argument and notation in [KS] as outlined in §1 we obtain

Proposition 2.7. *With W , N and g as above, and assuming the real Pontrjagin classes of M vanish,*

$$\begin{aligned} \text{ind} D_{(W \times N; g)}^+ &= \hat{A}(N) \left[\int_M d^{-1}(\hat{A} + a_{k-l}L)(p_*(M; g_M)) - a_{k-l}\eta(B(M, g_M)) - t(W) \right] \\ &\quad - \frac{h + \eta}{2}(M \times N; g_M + g_N), \end{aligned}$$

where L is the Hirzebruch L -polynomial, B denotes the signature operator,

$$a_n := 1/(2^{2n+1}(2^{2n-1} - 1)),$$

and the topological term $t(W)$ is given by

$$t(W) = -\left\langle (\hat{A} + a_{k-l}L)(j^{-1}p_*(W)), [W, M] \right\rangle + a_{k-l}\sigma(W)$$

where j denotes the inclusion map $j : H^*(W, M; \mathbb{R}) \rightarrow H^*(W; \mathbb{R})$ and $\sigma(W)$ is the signature of W .

Proof. It follows from Theorem 1.1 and the Atiyah-Patodi-Singer index theorem applied to the signature operator ([APS; 4.14]) that

$$\int_W (\hat{A} + a_{k-l}L)(p_*(W; g_W)) = \int_W \hat{A}(p_*(W, g_W)) + a_{k-l}\sigma(W) + a_{k-l}\eta(B(M, g_M)).$$

By [KS; Lemma 2.7] we have

$$\begin{aligned} \int_W (\hat{A} + a_{k-l}L)(p_*(W; g_W)) &= \int_M d^{-1}(\hat{A} + a_{k-l}L)(p_*(M; g_M)) \\ &\quad + \left\langle (\hat{A} + a_{k-l}L)(j^{-1}p_*(W)), [W, M] \right\rangle. \end{aligned}$$

Combining the above two statements with Corollary 2.6 yields the result. \square

If we assume that both g_M and $g_M + g_N$ have positive scalar curvature, (we can always achieve this by scaling g_M if necessary), then the term $h(M \times N; g_M + g_N)$ in the statement of Proposition 2.7 is zero. Collecting together the boundary terms as in [KS] then leads to

Definition 2.8. *Given M and N as above (so in particular the real Pontrjagin classes of M all vanish and $\hat{A}(N) \neq 0$), together with a positive scalar curvature g_M on M and a metric g_N on N such that the product metric $g_M + g_N$ on $M \times N$ has positive scalar curvature, we set*

$$\begin{aligned} \tilde{s}(M \times N, g_M + g_N) &:= \hat{A}(N) \left[\int_M d^{-1}(\hat{A} + a_{k-l}L)(p_*(M; g_M)) - a_{k-l}\eta(B(M, g_M)) \right] \\ &\quad - \frac{1}{2}\eta(D_{(M \times N; g_M + g_N)}). \end{aligned}$$

We now can write

$$\text{ind} D_{(W \times N; g)}^+ = \tilde{s}(M \times N, g_M + g_N) - \hat{A}(N)t(W).$$

Recalling the definition of the s -invariant (Definition 1.3) allows us to re-express \tilde{s} as

$$\tilde{s}(M \times N, g_M + g_N) = \hat{A}(N)s(M, g_M) + \frac{1}{2}\hat{A}(N)\eta(M, g_M) - \frac{1}{2}\eta(M \times N; g_M + g_N).$$

If the metric g on $W \times N$ has positive scalar curvature then the index term vanishes and we are left with

Lemma 2.9. *With all manifolds and metrics as above, if g is a positive scalar curvature metric on $W \times N$ (which as always is a product $dt^2 + g_M + g_N$ near the boundary) then*

$$\tilde{s}(M \times N, g_M + g_N) = \hat{A}(N)t(W).$$

The right-hand side of this expression depends only on the topology of $W \times N$, and is independent of the choice of metrics.

We now point out some key properties of \tilde{s} . The arguments needed here are essentially the same as those required to establish the equivalent properties for s . Although these arguments are for the most part suppressed in [KS], they are explained in depth in Chapter 5 of [TW], and we therefore omit the details here.

Lemma 2.10. *\tilde{s} is additive over disjoint unions in the following sense:*

$$\tilde{s}((M_1 \times N) \sqcup (M_2 \times N), g_{M_1} + g_N \sqcup g_{M_2} + g_N) = \tilde{s}(M_1 \times N, g_{M_1} + g_N) + \tilde{s}(M_2 \times N, g_{M_2} + g_N).$$

Lemma 2.11. *\tilde{s} is sensitive to the orientation of M in the sense that $\tilde{s}(M \times N, g_M + g_N) = -\tilde{s}((-M) \times N, g_M + g_N)$.*

Lemma 2.12. *\tilde{s} is additive over connected sums in the following sense:*

$$\tilde{s}((M_1 \sharp M_2) \times N, (g_{M_1} \sharp g_{M_2}) + g_N) = \tilde{s}(M_1 \times N, g_{M_1} + g_N) + \tilde{s}(M_2 \times N, g_{M_2} + g_N),$$

where $g_{M_1} \sharp g_{M_2}$ is the (canonical) Gromov-Lawson positive scalar curvature metric on the connected sum.

Proof of Theorem 0.3. Consider a path of positive scalar curvature metrics $g_{M \times N}(t)$ on $M \times N$ for $t \in [0, 1]$ say, where $g_{M \times N}(0)$ and $g_{M \times N}(1)$ are both product metrics with respect to the smooth product structure on $M \times N$. (Note that there is no need to assume that $g_{M \times N}(t)$ is a product metric for any $t \neq 0, 1$.) We first establish that $\tilde{s}(M \times N, g_{M \times N}(0)) = \tilde{s}(M \times N, g_{M \times N}(1))$.

It follows from a well-known observation about paths of positive scalar curvature metrics (see for example [Wr; Lemma 6.3]) that $g(t)$ can be adjusted to give a metric $g_{M \times N \times I}$ on $M \times N \times I$ for some interval I , which has positive scalar curvature globally, agrees with the metrics $g_{M \times N}(0)$ respectively $g_{M \times N}(1)$ when restricted to the two boundary components, and moreover is a product with respect to the t parameter near these boundary components. Thus taking $W = M \times I$, we see that by Lemma 2.9 we have

$$\tilde{s}(M \times N \sqcup (-M) \times N, g_{M \times N}(0) \sqcup g_{M \times N}(1)) = \hat{A}(N)t(M \times I).$$

By Lemmas 2.10 and 2.11 the left-hand side of this expression is equal to

$$\tilde{s}(M \times N, g_{M \times N}(0)) - \tilde{s}(M \times N, g_{M \times N}(1)).$$

We claim that $t(M \times I) = 0$. Now the $p_i(M \times I)$ vanish as the Pontrjagin classes of I and (by assumption) the Pontrjagin classes of M both vanish. Thus the $\langle (\hat{A} + a_{k-l}L)(\{j^{-1}p_i(M \times I)\}), [M \times I, \partial(M \times I)] \rangle$ term in $t(M \times I)$ must also be zero. It remains to show that

the signature $\sigma(M \times I) = 0$, but this follows since $M \times I \simeq M$ and so $H^{4k}(M \times I) = H^{4k}(M^{4k-1}) = 0$. Thus we have shown that \tilde{s} is an invariant of product metrics on $M \times N$ belonging to the same path-component of positive scalar curvature metrics.

Next, consider a spin structure preserving diffeomorphism $f : X \rightarrow M \times N$. We can use f to pull back a Riemannian product structure $(M \times N; g_M + g_N)$ to X . Let g be the resulting metric on X , so (X, g) is isometric to $(M \times N; g_M + g_N)$ via f . It is clear that (X, g) then has a Riemannian product structure $(X, g) = (M' \times N'; g_{M'} + g_{N'})$ for some submanifolds $M', N' \subset X$, with f restricting to give isometries $(M, g_M) \cong (M', g_{M'})$ and $(N, g_N) \cong (N', g_{N'})$. As each term on the right-hand side of the defining expression for $\tilde{s}(X, g)$ (Definition 2.8) is invariant under isometry, we see that \tilde{s} is invariant under the pull-back of Riemannian product structures by spin structure preserving diffeomorphisms.

Since $H^1(X; \mathbb{Z}_2) = 0$ by assumption, the spin structure corresponding to either orientation of X is unique. However a diffeomorphism $X \rightarrow X$ could reverse orientation, and therefore fail to preserve spin structures. In this case, though, we know from Lemma 2.11 that the sign of \tilde{s} changes. Thus we conclude that $|\tilde{s}|$ is invariant under the pull-back action of $\text{Diff}(X)$, and so gives a moduli space invariant.

Finally, we it remains to show that the moduli space has infinitely many path-components distinguished by \tilde{s} . The argument here is analogous to [KS; 2.15]. We note that by [Ca] there is a positive scalar curvature metric g on $S^{4(k-l)-1}$ which is extendable to a positive scalar curvature metric on a certain parallelisable bounding manifold (constructed by plumbing disc bundles). This bounding manifold has non-zero signature and vanishing Pontrjagin classes. It follows from Lemma 2.9 that $\tilde{s}(S^{4(k-l)-1} \times N, g + g_N)$ is a (non-zero) multiple of the (non-zero) signature of the bounding manifold. Consider the manifold

$$((M \# S_1^{4(k-l)-1} \# \dots \# S_p^{4(k-l)-1}) \times N, (g_M \# g \# \dots \# g) + g_N).$$

This is isometric to $(M \times N; g_p + g_N)$ for some positive scalar curvature metric g_p on M . Applying Lemma 2.9 to this latter manifold we obtain a different \tilde{s} -value for each $p \in \mathbb{N}$. Hence the result. \square

REFERENCES

- [APS] M. F. Atiyah, V.K. Patodi, I. M. Singer, *Spectral asymmetry and Riemannian geometry I*, Math. Proc. Camb. Phil. Soc. **77** (1975), 43-69.
- [Be] A.L. Besse, *Einstein Manifolds*, Springer-Verlag, Berlin (2002).
- [BERW] B. Botvinnik, J. Ebert, O. Randal-Williams, *Infinite loop spaces and positive scalar curvature*, arxiv:1411.7408.
- [BH] I. Belegradek and J. Hu, *Connectedness properties of the space of complete non-negatively curved planes*, Math. Ann. **362** (2015), 1273-1286. Erratum: Math. Ann. **364** (2016), 711-712.
- [BHSW] B. Botvinnik, B. Hanke, T. Schick, M. Walsh, *Homotopy groups of the moduli space of metrics of positive scalar curvature*, Geom. Topol. **14** (2010), 2047-2076.

- [CM] F. Codá Marques, *Deforming three-manifolds with positive scalar curvature*, Ann. of Math. (2) **176** (2012), no. 2, 815-863.
- [CS] D. Crowley, T. Schick, *The Gromoll filtration, KO-characteristic classes and metrics of positive scalar curvature*, Geom. Topol. **17** (2013), 1773-1790.
- [DKT] A. Dessai, S. Klaus, W. Tuschmann, *Nonconnected moduli spaces of nonnegative sectional curvature metrics on simply-connected manifolds*, arXiv 1601:04877.
- [FO1] F. T. Farrell, P. Ontaneda, *The Teichmüller space of pinched negatively curved metrics on a hyperbolic manifold is not contractible*, Ann. of Math. (2) **170** (2009), no. 1, 45-65.
- [FO2] F. T. Farrell, P. Ontaneda, *The moduli space of negatively curved metrics of a hyperbolic manifold*, J. Topol. **3** (2010), no. 3, 561-577.
- [FO3] F. T. Farrell, P. Ontaneda, *On the topology of the space of negatively curved metrics*, J. Diff. Geom. **86** (2010), no. 2, 273-301.
- [GL] M. Gromov, H.B. Lawson, *The classification of manifolds of positive scalar curvature*, Ann. Math. **111** (1980), 423-434.
- [Ha] A. Hatcher, *Algebraic Topology*, Cambridge University Press (2002).
- [HSS] B. Hanke, T. Schick, W. Steimle, *The space of metrics of positive scalar curvature*, Publ. Math. Inst. Hautes Études Sci. **120** (2014), 335-367.
- [KPT] V. Kapovitch, A. Petrunin, W. Tuschmann, *Non-negative pinching, moduli spaces and bundles with infinitely many souls*, J. Diff. Geom. **71** (2005) no. 3, 365-383.
- [KS] M. Kreck, S. Stolz, *Nonconnected moduli spaces of positive sectional curvature metrics*, J. Am. Math. Soc. **6** (1993), 825-850.
- [LM] H.B. Lawson, M.-L. Michelsohn, *Spin Geometry*, Princeton Math. Series **38**, Princeton University Press, (1989).
- [Mo] S. Morita, *Geometry of Differential Forms*, Translations of Mathematical Monographs vol. **201**, American Mathematical Society (2001).
- [MT] I. Madsen, J. Tornehave, *From Calculus to Cohomology*, Cambridge University Press (1997).
- [P] P. Petersen, *Riemannian Geometry*, Graduate Texts in Mathematics **171**, Springer-Verlag, (1998).
- [PS] P. Piazza, T. Schick, *Groups with torsion, bordism and rho-invariants*, Pacific J. Math. **232** (2007), no. 2, 355-378.
- [R] J. Rosenberg, *Manifolds of positive scalar curvature: a progress report*, Surveys in Differential Geometry vol. XI, International Press (2007), 259-294.
- [S] T. Schick, *The topology of scalar curvature*, arXiv:1405.4220v2.

- [TW] W. Tuschmann, D. J. Wraith, *Moduli spaces of Riemannian metrics*, Oberwolfach Seminars **46**, Birkhäuser, Springer Basel (2015).
- [Wa1] M. Walsh, *Cobordism invariance of the homotopy type of the space of positive scalar curvature metrics*, Proc. Amer. Math. Soc. **141** (2013), no. 7, 2475-2484.
- [Wa2] M. Walsh, *H-spaces, loop spaces and the space of positive scalar curvature metrics on the sphere*, Geom. Topol. **18** (2014), no. 4, 2189-2243.
- [Wr1] D. J. Wraith, *On the moduli space of positive Ricci curvature metrics on homotopy spheres*, Geom. Topol. **15** (2011), 1983-2015.
- [Wr2] D. J. Wraith, *Non-negative versus positive scalar curvature*, arXiv:1607.00657.

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